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A NOTE ON THE IDENTIFICATION OF LINEAR SYSTEMS

Richard Bellman

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SANTA MONICA • CALIFORNIA

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. This Memorandum considers the problem of ascertaining when certain conditions determine a unique solution of a linear system of differential equations.

SUMMARY

We consider the problem of ascertaining when the conditions

$$\int_0^T u(t)\varphi_j(t)dt = b_j, \quad j = 1,2,\ldots,N,$$

determine a unique solution of

$$u^{(N)} + a_1(t)u^{(N-1)} + \cdots + a_N(t)u = 0.$$

This is a particular problem arising in the general study of the identification of linear systems.

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A NOTE ON THE IDENTIFICATION OF LINEAR SYSTEMS

1. INTRODUCTION

Consider a linear system described by the function $u(t)$ satisfying the linear differential equation

$$(1.1) \qquad u^{(N)} + a_1(t)u^{(N-1)} + \dots + a_N(t)u = 0,$$

where the $a_i(t)$ are continuous functions of t in an interval $[0,T]$. The initial conditions, $u^{(i)}(0) = c_i$, $i = 0,1,\dots,N-1$, then determine the function $u(t)$ uniquely. The situation is, however, much more complicated, and hence more interesting, if we allow multipoint conditions of the type

$$(1.2) \qquad u(t_i) = c_i, \quad i = 1,2,\dots,N,$$

where $0 \le t_1 \le t_2 \le \dots \le t_N$. Coincidence of the t_i corresponds to conditions on the derivatives of $u(t)$. A very elegant result due to Polya [1] (see also Beckenbach and Bellman [2] for further discussion and references) resolves the problem of determining when (1.2) determines $u(t)$ for an arbitrary set of t-points.

From the standpoint of identification theory, we are asking that observation of the system at any N isolated times in the interval $[0,T]$ allow us to prescribe the behavior over the entire interval. There are many ways

of extending the scope of this problem. We can consider more general systems described by more general functional equations, such as differential-difference equations, partial differential equations, or nonlinear equations, or we can treat more general types of "observations."

We shall pursue the latter path here. In place of (1.2), consider the N conditions

$$(1.3) \qquad \int_0^T u(t)\varphi_j(t)dt = b_j, \quad j = 1,2,\ldots,N,$$

where the φ_j are given functions. We shall show that the solution of (1.1) is determined in this fashion, provided we impose certain conditions on $u(t)$ and the φ_j. These conditions are closely allied to the condition imposed by Polya, i.e., condition W.

2. PRELIMINARIES

Let $\{u_j(t)\}$, $j = 1,2,\ldots,N$, be N linearly independent solutions of (1.1). Write the general solution in the form

$$(2.1) \qquad u = \sum_{i=1}^{N} c_i u_i(t).$$

Then the integral conditions of (1.3) reduce to the system of linear algebraic equations

$$\sum_{i=1}^{N} c_i \int_0^T u_i(t)\varphi_j(t)dt = b_j, \quad j = 1,2,\ldots,N. \tag{2.2}$$

The determination of a unique function $u(t)$ satisfying both (1.1) and (1.3) has thus been converted into an examination of the nonvanishing of the determinant

$$\left|\int_0^T u_i(t)\varphi_j(t)dt\right|, \quad i,j = 1,2,\ldots,N. \tag{2.3}$$

To study this question we invoke an extremely useful determinantal identity

$$\begin{aligned}\left|\int_0^T u_i(t)\varphi_j(t)dt\right| \\ = \frac{1}{N!}\int_0^T\cdots\int_0^T P(t_1,t_2,\ldots,t_N,u)P(t_1,t_2,\ldots,t_N,\varphi)dt_1dt_2\cdots dt_N,\end{aligned} \tag{2.4}$$

where

$$P(t_1,t_2,\ldots,t_N,u) = \begin{vmatrix} u_1(t_1) & u_1(t_2) & \cdots & u_1(t_N) \\ u_2(t_1) & u_2(t_2) & \cdots & u_2(t_N) \\ \vdots & & & \\ u_N(t_1) & u_N(t_2) & \cdots & u_N(t_N) \end{vmatrix}, \tag{2.5}$$

the Polyan of the N functions $u_1(t),u_2(t),\ldots,u_N(t)$.

In the case $N = 2$, the symmetry of the integrand in (2.4) about the line $t_1 = t_2$ permits us readily to conclude that

$$(2.6) \qquad \left|\int_0^T u_i(t)\varphi_j(t)dt\right| = \int\int_{0\le t_1\le t_2\le T} P(t_1,t_2,u)P(t_1,t_2,\varphi)dt_1dt_2.$$

In general, we can conclude that

$$(2.7) \qquad \left|\int_0^T u_i(t)q_j(t)dt\right| = \int\cdots\int_{0\le t_1\le t_2\le\cdots\le t_N\le T} P(t_1,\ldots,t_N,u)P(t_1,\ldots,t_N,\varphi)dt_1dt_2\cdots dt_N.$$

This relation is the key to what follows.

3. CEBYCEV SYSTEMS

Let $\{u_i(t)\}$, $i = 1,2,\ldots,N$, be a set of N linearly independent continuous functions. We say that it is a Cebycev system in $[0,T]$ if every linear combination, $\sum_i c_i^2 \neq 0$,

$$(3.1) \qquad f(t) = \sum_{i=1}^N c_i u_i(t),$$

has at most N zeros in $[0,T]$. A basic result (see Gantmacher-Krein [3]) is that a necessary and sufficient condition for $\{u_i\}$ to be a Cebycev system is that

$$(3.2) \qquad P(t_1,t_2,\ldots,t_N,u) \neq 0$$

for any set of t_i in $[0,T]$ satisfying the condition

$$(3.3) \qquad 0 \le t_1 < t_2 < \cdots < t_N \le T.$$

It follows that P has the same sign in this region.

Examples of Cebycev systems are $\{t^i\}$, $i = 0,1,\ldots,N-1$, $\{e^{\lambda_i t}\}$, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_N$. Sometimes the oscillation condition is easy to verify, and at other times, the determinantal condition.

4. A SUFFICIENT CONDITION FOR IDENTIFICATION

Using the foregoing results, we can assert:

Theorem. If any particular set of N linearly independent solutions of (1.1) is a Cebycev set in $[0,T]$ (and hence every such set), and if the φ_j are a Cebycev set, then the conditions of (1.3) determine $u(t)$ uniquely.

It is clear that this condition on the solutions of $L(u) = 0$ is a characteristic value condition. For example,

$$(4.1) \qquad u'' + u = 0$$

satisfies the condition in $[0,T]$ if $T < 2\pi$, but not for $T \ge 2\pi$, as $u = \sin t$ shows. In general, we can assert that $L(u) = 0$ satisfies the condition if T is sufficiently small.

REFERENCES

1. Polya, G., "On the Mean-value Theorem Corresponding to a Given Linear Homogeneous Differential Equation," Trans. Amer. Math. Soc., Vol. 24, 1922, pp. 312–324.

2. Beckenbach, E. F., and R. Bellman, Inequalities, Springer, Berlin, 1961.

3. Gantmacher, F. R., and M. G. Krein, Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme, Akademie Verlag, Berlin, 1960.

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